Optimal Strong Parity Edge-Coloring of Complete Graphs

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Abstract

A *parity walk* in an edge-coloring of a graph is a walk along which each color is used an even number of times. Let $p(G)$ be the least number of colors in an edge-coloring of G having no parity path (a *parity edge-coloring*). Let $\widehat{p}(G)$ be the least number of colors in an edge-coloring of G having no open parity walk (a *strong parity edge-coloring*). Always $\widehat{p}(G) \geq p(G) \geq \chi'(G)$. We prove that $\widehat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n. The optimal strong parity edge-coloring of K_n is unique when n is a power of 2, and the optimal colorings are completely described for all n.

1 Introduction

Our work began by studying which graphs embed in the hypercube Q_k, the graph with vertex set $\{0,1\}^k$ in which vertices are adjacent when they differ in exactly one coordinate. Coloring each edge with the position of the bit in which its endpoints differ yields two necessary conditions for the coloring inherited by a subgraph G:

1) every cycle uses each color an even number of times,

2) every path uses some color an odd number of times.

Existence of a k-edge-coloring satisfying conditions (1) and (2) is also sufficient for a connected graph G to be a subgraph of Q_k. This characterization of subgraphs of Q_k appeared in 1972 (Havel and Morávek [8]). The problem was studied as early as 1953 (Shapiro [13]).

Let the *usage* of a color on a walk be the parity of the number of times it appears along the walk. A *parity walk* is a walk in which the usage of every color is even. Condition (1) above states that every cycle is a parity walk, and (2) states that no path is a parity walk.

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In general, a *parity edge-coloring* is an edge-coloring with no parity path, and a *strong parity edge-coloring (spec)* is an edge-coloring with no open parity walk (that is, every parity walk is closed). Some graphs embed in no hypercube, but giving the edges distinct colors produces a spec for any graph. Hence the *parity edge-chromatic number* $p(G)$ and the *strong parity edge-chromatic number* $\widehat{p}(G)$, defined respectively to be the minimum numbers of colors in a parity edge-coloring of G and in a spec of G, are well defined. Elementary results on these parameters appear in [5].

When T is a tree, $\widehat{p}(T) = p(T) = k$, where k is the least integer such that T embeds in Q_k [5]. Since incident edges of the same color would form a parity path of length 2, every parity edge-coloring is a proper edge-coloring, and hence $p(G) \geq \chi'(G)$, where $\chi'(G)$ denotes the edge-chromatic number. Although there are graphs G with $\widehat{p}(G) > p(G)$ [5], it remains unknown how large $\widehat{p}(G)$ can be when $p(G) = k$. It also remains unknown whether there is a bipartite graph G with $\widehat{p}(G) > p(G)$.

When n is a power of 2, we will prove that the complete graph K_n has a unique optimal spec (up to isomorphism), which will help us determine $\widehat{p}(K_n)$ for all n. With a suitable naming of the vertices, we call this edge-coloring of K_n the "canonical" coloring.

Definition 1.1 For $A \subseteq \mathbb{F}_2^k$, let $K(A)$ be the complete graph with vertex set A. The *canonical coloring* of $K(A)$ is the edge-coloring f defined by $f(uv) = u + v$, where $u + v$ is binary vector addition. When $n = 2^k$, letting $A = \mathbb{F}_2^k$ yields the canonical coloring of K_n.

Lemma 1.2 *For $A \subseteq \mathbb{F}_2^k$, the canonical coloring of $K(A)$ is a spec. Consequently, if $n = 2^k$, then $\widehat{p}(K_n) = p(K_n) = \chi'(K_n) = n - 1$.*

Proof. If W is an open walk, then its endpoints differ in some bit i. Thus in the canonical coloring the total usage of colors flipping bit i along W is odd, and hence some color has odd usage on W. The canonical coloring of $K(\mathbb{F}_2^k)$ uses $2^k - 1$ colors (the color 0^k is not used). The lower bound follows from $\widehat{p}(G) \geq p(G) \geq \chi'(G) \geq \Delta(G)$. ∎

Since every complete graph is a subgraph of the next larger complete graph, we obtain $\widehat{p}(K_n) \leq 2^{\lceil \lg n \rceil} - 1$. In Section 2, we prove that this upper bound is exact. Our proof is expressed using linear subspaces of binary vector spaces.

Parity edge-coloring relates to a less restrictive problem. A walk of length $2k$ is *repetitive* if the ith and $(k+i)$th edges have the same color, for $1 \leq i \leq k$. A *Thue coloring* is an edge-coloring with no repetitive path, and the *Thue number* $t(G)$ is the minimum number of colors in a Thue coloring of G. Every parity edge-coloring is a Thue coloring, so $t(G) \leq p(G)$. Alon, Grytczuk, Hałuszczak, and Riordan [2] observed that the canonical coloring yields $t(K_n) \leq 2^{\lceil \lg n \rceil} - 1$. It seems no good lower bounds on $t(K_n)$ are known. By our result, a Thue coloring of K_n better than the canonical coloring must contain an open parity walk.

To further motivate our focus on complete graphs, we show that our main result strengthens a special case of Yuzvinsky's Theorem on sums of binary vectors. To state it, we need the Hopf–Stiefel function from the theory of quadratic forms.

Definition 1.3 (Hopf [9], Stiefel [14]) For positive integers r and s, define $r \circ s$ to be the least integer n such that $(x+y)^n$ is in the ideal of $\mathbb{F}_2[x,y]$ generated by x^r and y^s.

In non-algebraic language, the definition has the following equivalent phrasing: $r \circ s$ is the least n such that $\binom{n}{k}$ is even for each k with $n-s < k < r$. The condition becomes vacuous if $n \geq r+s-1$, so trivially $r \circ s \leq r+s-1$.

Theorem 1.4 (Yuzvinsky [15]) *For $A, B \subseteq \mathbb{F}_2^k$, let $C = \{a+b : a \in A, b \in B\}$. If $|A| = r$ and $|B| = s$, then $|C| \geq r \circ s$.*

Generalizations and alternative proofs of Yuzvinsky's Theorem appear in [1], [4], [6]. The theorem is related to our results via a simple formula for the Hopf–Stiefel function recently proved by Plagne [12]. Subsequently, Károlyi [10] gave a short inductive proof. See [7] for a thorough survey of alternative formulas, related results, and generalizations.

Theorem 1.5 (Plagne [12], Károlyi [10]) $r \circ s = \min_{k \in \mathbb{N}} \left\{ 2^k \left(\left\lceil \frac{r}{2^k} \right\rceil + \left\lceil \frac{s}{2^k} \right\rceil - 1 \right) \right\}$.

When $A = B$ and both have size r, the minimization yields $r \circ r = 2^{\lceil \lg r \rceil}$. Yuzvinsky's Theorem for this case says that every canonical coloring of K_r uses at least $2^{\lceil \lg r \rceil} - 1$ colors. Our result shows that in the more general family of strong parity edge-colorings, it remains true that at least $2^{\lceil \lg r \rceil} - 1$ colors are needed.

The canonical coloring extends to complete bipartite graphs in a natural way: if $A, B \subseteq \mathbb{F}_2^k$ and $K(A,B)$ is the complete bipartite graph with partite sets A and B, then the edge-coloring defined by $f(ab) = a+b$ is a spec. The bound in Yuzvinsky's Theorem is always tight (see [6]); that is, for $r, s \leq 2^k$ there exist $A, B \subseteq \mathbb{F}_2^k$ with $|A| = r$, $|B| = s$, and $|C| = r \circ s$. Consequently, $\widehat{p}(K_{r,s}) \leq r \circ s$. We conjecture that equality holds. A direct proof in the graph-theoretic setting would strengthen all cases of Yuzvinsky's Theorem.

Conjecture 1.6 $\widehat{p}(K_{r,s}) = r \circ s$.

2 The Lower Bound

In this section, we use the closure of linear spaces under addition to prove that $\widehat{p}(K_n) \geq 2^{\lceil \lg n \rceil} - 1$. The main idea is to introduce an additional vertex without needing additional colors until a power of 2 is reached. We begin by proving that every optimal spec of K_n is a canonical coloring when n is a power of 2.

Definition 2.1 An edge-coloring f of a graph G satisfies the *4-constraint* if whenever $f(uv) = f(xy)$ and $vx \in E(G)$, also $uy \in E(G)$ and $f(uy) = f(vx)$.

Lemma 2.2 *If f is a parity edge-coloring in which every color class is a perfect matching, then f satisfies the 4-constraint.*

Proof. Otherwise, given $f(uv) = f(xy)$, the edge of color $f(vx)$ incident to u forms a parity path of length 4 with uv, vx, and xy. ∎

Theorem 2.3 *If f is a parity edge-coloring of K_n in which every color class is a perfect matching, then f is a canonical coloring and n is a power of 2.*

Proof. Every edge is a canonically colored K_2. Let R be a largest vertex set such that $|R|$ is a power of 2 and f restricts to a canonical coloring on R. Define j by $|R| = 2^{j-1}$. Let ϕ be a bijection from R to $\mathbb{F}_2^{j-1}$ under which f is the canonical coloring. We prove the claim by showing that if $|R| < n$, then f canonically colors a subgraph twice as big.

Since f is canonical on R, every color used within R by f is used on a perfect matching of R. The vertices of R have other neighbors in K_n, so there is a color c not used within R. Since c is used on a perfect matching, c matches R to some set U. Let $R' = R \cup U$. Define $\phi'\colon R' \to \mathbb{F}_2^j$ as follows: for $x \in R$, obtain $\phi'(x)$ by appending 0 to $\phi(x)$; for $x \in U$ obtain $\phi'(x)$ by appending 1 to $\phi(x')$, where x' is the neighbor of x in color c. Within R', we henceforth refer to the vertices by their names under ϕ'.

By Lemma 2.2, the 4-constraint holds for f. The 4-constraint copies the coloring from the edges within R to the edges within U. That is, consider $x', y' \in U$ arising from $x, y \in R$, with $f(xx') = f(yy') = c$. Now $f(x'y') = f(xy) = x + y = x' + y'$, using the 4-constraint, the fact that f is canonical on R, and the definition of ϕ'. Hence f is canonical within U.

Finally, let u be the name of the color on the edge $0^j u$, for $u \in U$. For any $v \in R$, let $w = u + v$; note that $w \in U$. Both $0^j v$ and uw have color v, since f is canonical within R and within U. By applying the 4-constraint to $\{v0^j, 0^j w, wu\}$, we conclude that $f(uv) = f(0^j w) = w$. Since $w = u + v$, this completes the proof that f is canonical on R'. ∎

For a proper edge-coloring of K_n, the 4-constraint is equivalent to the property that the six edges on any four vertices receive three colors or six colors. Independently of our proof, Keevash and Sudakov [11] proved that if that property holds for a proper edge-coloring with $n-1$ colors (one where every color class is a perfect matching), then n is a power of 2. They also observed that the canonical coloring has this property. Their proof appears also in [3].

In connection with the uniqueness result, Mubayi asked whether a stability property holds. That is, when n is a power of 2, does there exist a parity edge-coloring or a spec of K_n that has only $(1 + o(1))n$ colors but is "far" from the canonical coloring?

The main result needs several algebraic observations. Relative to any k-edge-coloring f, the *parity vector* $\pi(W)$ of a walk W is the binary k-tuple whose ith bit agrees in parity with the usage of color i along W. Let the *parity space* L_f be the set of parity vectors of closed walks. We note that L_f is a linear subspace of $\mathbb{F}_2^k$.

Lemma 2.4 *If f is an edge-coloring of a connected graph G, then L_f is a binary vector space.*

Proof. Since $L_f \subseteq \mathbb{F}_2^k$, it suffices to show that L is closed under addition. Given a u,u-walk W and a v,v-walk W', let P be a u,v-path in G, and let $\overline{P}$ be its reverse. Following $W, P, W', \overline{P}$ in succession yields a u,u-walk with parity vector $\pi(W)+\pi(W')$. ∎

For a vector space L, let $w(L)$ be the least number of nonzero coordinates of a nonzero vector in L (set $w(L)=\infty$ if $L=\{0\}$). For an edge-coloring f of K_n, $w(L_f)$ determines whether f is a spec. Let the *weight* of a vector in $\mathbb{F}_2^k$ be the number of nonzero entries.

Lemma 2.5 *If an edge-coloring f of a graph G is a spec, then $w(L_f) \geq 2$. The converse holds when $G = K_n$.*

Proof. If the parity vector of a closed walk W has weight 1, then one color has odd usage in W (say on edge e). Now $W-e$ is an open parity walk, and f is not a spec.

If f is not a spec, then $\pi(W')=0$ for some open walk W'. In K_n, the ends of W' are adjacent, and adding that edge yields a closed walk whose parity vector has weight 1. ∎

Lemma 2.6 *For any colors a and b in an optimal spec f of K_n, there is some closed walk W on which the colors having odd usage are a, b, and one other.*

Proof. We use Lemma 2.5 repeatedly. Since f is optimal, merging the colors a and b into a single color a' yields an edge-coloring f' that is not a spec. Hence under f' there is a closed walk W on which f' has odd usage for only one color c. Also $c \neq a'$, since otherwise f has odd usage on W for only a or b. With $c \neq a'$ and the fact that f has odd usage for at least two colors on W, both a and b also have odd usage on W, and W is the desired walk. ∎

The same idea as in Lemma 2.6 shows that $w(L_f) \geq 3$ when f is an optimal spec of K_n, but we do not need this observation. We note, however, that the condition $w(L_f) \geq 3$ is the condition for L_f to be the set of codewords for a 1-error-correcting code. Indeed, when $n = 2^k$ and f is the canonical coloring, L_f is a perfect 1-error-correcting code of length $n-1$.

A *dominating vertex* in a graph is a vertex adjacent to all others. We use $d_H(v)$ and $N_H(v)$ to denote the degree and neighborhood of a vertex v in a graph H.

Lemma 2.7 *If f is an edge-coloring of a graph G with a dominating vertex v, then L_f is the span of the parity vectors of triangles containing v.*

Proof. By definition, the span is contained in L_f. Conversely, consider any $\pi(W) \in L_f$. Let S be the set of edges with odd usage in W, and let H be the spanning subgraph of G with edge set S. Since the total usage at each vertex of W is even, H is an even subgraph of G. Hence H decomposes into cycles, which are closed walks, and $\pi(W)$ is the sum of the parity vectors of these cycles.

It therefore suffices to show that S is the set of edges that appear in an odd number of the triangles formed by v with edges of $H-v$. Each edge of $H-v$ is in one such triangle, so

we need consider only edges involving v. An edge vw lies in an odd number of these triangles if and only if $d_{H-v}(w)$ is odd, which occurs if and only if $w \in N_H(v)$, since $d_H(w)$ is even. By definition, $vw \in E(H)$ if and only if vw has odd usage in W and hence lies in S. ∎

Lemma 2.8 *If f is an optimal spec of K_n that uses some color fewer than $n/2$ times, then f extends to a spec of K_{n+1} using the same colors.*

Proof. View K_{n+1} as arising from K_n by adding a vertex u. Let a be a color used fewer than $n/2$ times by f, and let v be a vertex of K_n at which a does not appear.

We use f to define f' on $E(K_{n+1})$. Let f' agree with f on $E(K_n)$, and let $f'(uv) = a$. To define f' on each remaining edge uw, first let $b = f(vw)$. By Lemma 2.6, there is a closed walk W with odd usage precisely for a and b and some third color c under f. Let $f'(uw) = c$.

Note that f' uses the same colors as f. It remains only to show that f' is a spec. To do this we prove that $w(L_{f'}) \geq 2$, by showing that $L_{f'} \subseteq L_f$. By Lemma 2.7, it suffices to show that $\pi(T) \in L_f$ when T is a triangle in K_{n+1} containing v.

Triangles not containing u lie in the original graph and have parity vectors in L_f. Hence we consider the triangle T formed by $\{u, v, w\}$. Now $\pi(T) = \pi(W) \in L_f$, where W is the walk used to specify $f'(uw)$. ∎

Theorem 2.9 $\widehat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$.

Proof. If some color class in an optimal spec is not a perfect matching, then $\widehat{p}(K_n) = \widehat{p}(K_{n+1})$, by Lemma 2.8. This vertex absorption cannot stop before the number of vertices reaches a power of 2, because when every color class is a perfect matching the coloring is canonical, by Theorem 2.3. It cannot continue past $2^{\lceil \lg n \rceil}$ vertices, since the maximum degree is a lower bound on χ' and $\widehat{p}$. Hence $\widehat{p}(K_n) = \widehat{p}(K_{2^{\lceil \lg n \rceil}}) = 2^{\lceil \lg n \rceil} - 1$. ∎

Corollary 2.10 *If f is an optimal spec of K_n, then f is obtained by deleting vertices from the canonical coloring of $K_{2^{\lceil \lg n \rceil}}$.*

Proof. By Lemma 2.8, we may extend f to an optimal spec f' of $K_{2^{\lceil \lg n \rceil}}$; by Theorem 2.3, f' is the canonical coloring. ∎

One may ask whether every edge-coloring of K_n that satisfies the 4-constraint is a spec or a parity edge-coloring. Examples that show the answer is no. Similarly, not every parity edge-coloring of K_n is a spec. Nevertheless, it may be that every *optimal* parity edge-coloring is a spec. We offer the following conjecture, which in [5] we proved for $n \leq 16$.

Conjecture 2.11 $p(K_n) = \widehat{p}(K_n)$ *for every positive integer n.*

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