Parity and Strong Parity Edge-Coloring of Graphs

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Abstract

A *parity walk* in an edge-coloring of a graph is a walk traversing each color an even number of times. We introduce two parameters. Let $p(G)$ be the least number of colors in a *parity edge-coloring* of G (a coloring having no parity path). Let $\widehat{p}(G)$ be the least number of colors in a *strong parity edge-coloring* of G (a coloring having no open parity walk). Note that $\widehat{p}(G) \geq p(G) \geq \chi'(G)$.

The values $p(G)$ and $\widehat{p}(G)$ may be equal or differ, with equality conjectured for all bipartite graphs. If G is connected, then $p(G) \geq \lceil \lg |V(G)| \rceil$, with equality for paths and even cycles (C_n needs one more color for odd n). The proof that $\widehat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n will appear later; the conjecture that $p(K_n) = \widehat{p}(K_n)$ is proved here for $n \leq 16$ and other cases. Also, $p(K_{2,n}) = \widehat{p}(K_{2,n}) = 2\lceil n/2 \rceil$. In general, $\widehat{p}(K_{m,n}) \leq m' \lceil n/m' \rceil$, where $m' = 2^{\lceil \lg m \rceil}$. We compare these to other parameters and pose many open questions.

1 Introduction

We began by studying which graphs embed in the hypercube Q_k, the graph with vertex set $\{0,1\}^k$ in which vertices are adjacent when they differ in exactly one position. Coloring each edge with the position of the bit where its endpoints differ satisfies two properties on each subgraph G:

1) every cycle uses each color an even number of times,

2) every path uses some color an odd number of times.

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In somewhat different language, Havel and Movárek [12] proved in 1972 that the existence of such an edge-coloring with k colors characterizes the connected graphs that are subgraphs of Q_k. The problem was studied as early as 1953 by Shapiro [24].

Parity of the usage of colors along walks suggests two edge-coloring parameters that have interesting properties and applications. Let the *usage* of a color on a walk be the parity of the number of times it appears along the walk. A *parity walk* is a walk in which every color has even usage.

Let a *parity edge-coloring* (pec) be an edge-coloring having no parity path. Using distinct colors on all edges produces a parity edge-coloring. Hence we introduce the *parity edge-chromatic number* $p(G)$ to be the minimum number of colors in a pec of G. Paths of length 2 force $p(G) \geq \chi'(G)$, where $\chi'(G)$ is the edge-chromatic number.

A more restricted notion has better algebraic properties. A *strong parity edge-coloring (spec)* is an edge-coloring in which every parity walk is closed. Using distinct colors again works, so we let the *strong parity edge-chromatic number* $\widehat{p}(G)$ be the minimum number of colors in a spec. A spec has no parity path, so every spec is a pec, and always $\widehat{p}(G) \geq p(G)$.

Characterizing subgraphs of Q_k using parity edge-coloring yields $p(G) \geq \lceil \lg |V(G)| \rceil$ when G is connected, with equality for a path or even cycle (throughout, lg denotes $\log_2$). When n is odd, $p(C_n) = \widehat{p}(C_n) = 1 + \lceil \lg n \rceil$. Also $p(K_{2,n}) = \widehat{p}(K_{2,n}) = 2\lceil n/2 \rceil$. In these examples, $p(G) = \widehat{p}(G)$; we also give examples where equality fails.

In this paper, we primarily explore the elementary properties of these parameters. In a subsequent paper [2], we prove that $\widehat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$. This strengthens a special case of a theorem of Yuzvinsky about sums of binary vectors (see Section 3).

Among the questions we raise in Section 6 is whether also $p(K_n) = 2^{\lceil \lg n \rceil} - 1$; we prove this for $n \leq 16$. The complete bipartite graph $K_{n,n}$ behaves like K_n in that $p(K_{n,n}) = \widehat{p}(K_{n,n}) = \chi'(K_{n,n}) = n$ when $n = 2^k$. Also, $\widehat{p}(K_{n,n}) \leq \widehat{p}(K_n) + 1$ for all n; we conjecture that equality holds. We show that $\widehat{p}(K_{m,n}) \leq m' \lceil n/m' \rceil$, where $m \leq n$ and $m' = 2^{\lceil \lg m \rceil}$.

As a possible tool for exploring conjectured equalities between p and $\widehat{p}$, we introduce a generalization. A *parity r-set edge-coloring* assigns r colors to each edge so that every selection of one color from the set at each edge yields a parity edge-coloring. Let $p_r(G)$ be the minimum number of colors used. Always $p_r(G) \leq rp(G)$, and we prove equality for paths. Proving $p_2(K_n) = 2p(K_n)$ could be a step toward proving $p(K_n) = 2^{\lceil \lg n \rceil} - 1$.

In Section 5, we distinguish parity edge-coloring from related edge-coloring problems. Section 6 poses many open questions.

2 Elementary Properties and Examples

First we formalize elementary observations from the Introduction.

Remark 2.1 *For every graph G, $\widehat{p}(G) \geq p(G) \geq \chi'(G)$, and the parameters $\widehat{p}$ and p are monotone under the subgraph relation.*

Proof. We have $p(G) \geq \chi'(G)$ by considering paths of length 2, and $\widehat{p}(G) \geq p(G)$ since closed walks are not paths. For $H \subseteq G$, a pec or spec of G restricts to such an edge-coloring on H, since every parity walk in the restriction to H is a parity walk in the coloring on G. ■

When G is a forest, every pec is also a spec, so $p(G) = \widehat{p}(G)$. Edge-coloring the hypercube by coordinates shows that $p(Q_k) \leq \widehat{p}(Q_k) \leq k$. Hence $p(G) \leq k$ if $G \subseteq Q_k$. For trees, we prove the converse.

Given a k-edge-coloring f and a walk W, we use $\pi(W)$ to denote the parity vector of W, recording the usage of each color as 0 or 1. When walks W and W' are concatenated, the parity vector of the concatenation is the vector binary sum $\pi(W) + \pi(W')$. The *weight* of a vector is the number of nonzero positions.

Theorem 2.2 *A tree T embeds in the k-dimensional hypercube Q_k if and only if $p(T) \leq k$.*

Proof. We have observed necessity. Conversely, let f be a parity k-edge-coloring of T (there may be unused colors if $p(T) < k$). Fix a root vertex r in T. Define $\phi\colon V(T) \to V(Q_k)$ by setting $\phi(v) = \pi(W)$, where W is the r, v-path in T.

When $uv \in E(T)$, the r, u-path and r, v-path in T differ in one edge, so $\phi(u)$ and $\phi(v)$ are adjacent in Q_k. It remains only to check that ϕ is injective. The parity vector for the u, v-path P in T is $\phi(u) + \phi(v)$, since summing the r, u-path and r, v-path cancels the portion from r to P. Since f is a parity edge-coloring, $\phi(P)$ is nonzero, and hence $\phi(u) \neq \phi(v)$. ■

When k is part of the input, recognizing subgraphs of Q_k is NP-complete [16], and this remains true when the input is restricted to trees [25]. Therefore, computing $p(G)$ or $\widehat{p}(G)$ is NP-hard even when G is a tree. Perhaps there is a polynomial-time algorithm for trees with bounded degree or bounded diameter.

The Havel–Movárek characterization of subgraphs of Q_k follows easily from Theorem 2.2 (they also proved statements equivalent to Theorem 2.2 and Corollary 2.5.) Their proof is essentially the same as ours, but our organization is different in the language of pecs.

Corollary 2.3 *A graph G is a subgraph of Q_k if and only if G has a parity k-edge-coloring in which every cycle is a parity walk.*

Proof. We have observed necessity. For sufficiency, choose a spanning tree T. Since $p(T) \leq p(G) \leq k$, Theorem 2.2 implies that $T \subseteq Q_k$. Map T into Q_k using ϕ as defined in the proof of Theorem 2.2. For each $xy \in E(G) - E(T)$, the cycle formed by adding xy to T is given to be a parity walk. Hence the x, y-path in T has parity vector with weight 1. This makes $\phi(x)$ and $\phi(y)$ adjacent in Q_k, as desired. ■

Mitas and Reuter [20] later gave a lengthy proof motivated by studying subdiagrams of the subset lattice. They also characterized the graphs occurring as induced subgraphs of Q_k as those having a k-edge-coloring satisfying our properties (1) and (2) and a third property stating essentially that if the parity vector of a walk has weight 1, then its endpoints are adjacent.

Spanning trees yield a general lower bound on $p(G)$, which holds with equality for paths, even cycles, and connected spanning subgraphs of Q_k.

Corollary 2.4 *If G is connected, then $p(G) \geq \lceil \lg n(G) \rceil$.*

Proof. If T is a spanning tree of G, then $p(G) \geq p(T)$. Since T embeds in the hypercube of dimension $p(T)$, we have $n(G) = n(T) \leq 2^{p(T)} \leq 2^{p(G)}$. ■

Corollary 2.5 *For all n, $p(P_n) = \widehat{p}(P_n) = \lceil \lg n \rceil$. For even n, $p(C_n) = \widehat{p}(C_n) = \lceil \lg n \rceil$.*

Proof. The lower bounds follow from Corollary 2.4. The upper bounds hold because Q_k contains cycles of all even lengths up to 2^k. ■

A result equivalent to $p(P_n) = \widehat{p}(P_n) = \lceil \lg n \rceil$ appears in [12] (without defining either parameter). When n is odd, $p(C_n) = \widehat{p}(C)n) = \lceil \lg n \rceil + 1$. To prove this, we begin with simple observations about adding an edge.

Lemma 2.6 *(a) If e is an edge in a graph G, then $p(G) \leq p(G - e) + 1$. (b) If also $G - e$ is connected, then $\widehat{p}(G) \leq \widehat{p}(G - e) + 1$.*

Proof. (a) Put an optimal parity edge-coloring on $G - e$ and add a new color on e. There is no parity path avoiding e, and any path through e uses the new color exactly once.

(b) Put an optimal spec on $G - e$ and add a new color on e. Let P be a u, v-path in $G - e$, where u and v are the endpoints of e. Suppose that there is an open parity walk W. Note that W traverses e an even number of times, since no other edge has the same color as e. Form W' by replacing each traversal of e by P or its reverse, depending on the direction of traversal of e. Every edge is used with the same parity in W' and W, and the endpoints are unchanged, so W' is an open parity walk in $G - e$. This is a contradiction. ■

Theorem 2.7 *If n is odd, then $p(C_n) = \widehat{p}(C_n) = \lceil \lg n \rceil + 1$.*

Proof. Lemma 2.6(b) yields the upper bound, since $\widehat{p}(P_n) = \lceil \lg n \rceil$.

For the lower bound, we show first that $\widehat{p}(C_n) = p(C_n)$ (this and Lemma 2.6(a) yield an alternative proof of the upper bound). Let W be an open walk, and let W' be the subgraph formed by the edges with odd usage in W. The sum of the usage by W of edges incident to a vertex x is odd if and only if x is an endpoint of W. Hence W' has odd degree precisely at the endpoints of W. Within C_n, this requires W' to be a path P joining the endpoints of W. Under a parity edge-coloring f, some color has odd usage along P, and this color has odd usage in W. Hence f has no open parity walk, and every parity edge-coloring is a spec.

It now suffices to show that $\widehat{p}(C_n) \geq p(P_{2n})$. Given a spec f of C_n, we form a parity edge-coloring g of P_{2n} with the same number of colors. Let $v_1, \ldots, v_n$ be the vertices of C_n in order, and let $u_1, \ldots, u_n, w_1, \ldots, w_n$ be the vertices of P_{2n} in order. Define g by letting $g(u_i u_{i+1}) = g(w_i w_{i+1}) = f(v_i v_{i+1})$ for $1 \leq i \leq n-1$ and letting $g(u_n w_1) = f(v_n v_1)$.

Each path in P_{2n} corresponds to an open walk in C_n or to one trip around the cycle. There is no parity path of the first type, since f is a spec. There is none of the second type, since C_n has odd length. ■

The "unrolling" technique of Theorem 2.7 leads to an example G with $\widehat{p}(G) > p(G)$, which easily extends to generate infinite families.

Example 2.8 Form a graph G by identifying a vertex of K_3 with an endpoint of P_8. Since $p(K_3) = p(P_7) = 3$, adding the connecting edge yields $p(G) \leq 4$ (see Lemma 2.6(a)).

We claim that $\widehat{p}(G) \geq p(P_{18}) = 5$ (this shows also that Lemma 2.6(b) may fail when $G - e$ is disconnected). We copy a spec f of G onto P_{18} with the path edges doubled. Beginning with the vertex of degree 1 in G, walk down the path, once around the triangle, and back up the path. This walk has length 17; copy the colors of its edges in order to the edges of P_{18} in order to form an edge-coloring g of P_{18}.

Each path in P_{18} corresponds to an open walk in G or a closed walk that traverses the triangle once. There is no parity path of the first type, since f is a spec. There is none of the second type, since such a closed walk has odd length. This proves the claim.

Since $\widehat{p}(K_3) = \widehat{p}(P_7) = 3$, this graph G also shows that adding an edge can change $\widehat{p}$ by more than 1 when G is disconnected.

We know of no bipartite graph G with $\widehat{p}(G) > p(G)$. Nevertheless, not every optimal parity edge-coloring of a bipartite graph is a spec.

Example 2.9 Let G be the graph obtained from C_6 by adding two pendant edges at one vertex. Let W be the spanning walk that starts at one pendant vertex, traverses the cycle, and ends at the other pendant vertex. Let f be the 4-edge-coloring that colors the edges of W in order as a, b, a, c, b, d, c, d. Although f is an optimal parity edge-coloring ($\Delta(G) = 4$), it uses each color twice on the open walk W, so it is not a spec. Changing the edge of color d on the cycle to color a yields a strong parity 4-edge-coloring.

3 Specs and Canonical Colorings

The monotonicity of p and $\widehat{p}$ guarantees that K_n has the largest value of both parameters among all n-vertex graphs. Thus determining $p(K_n)$ or $\widehat{p}(K_n)$ solves the corresponding extremal problem for n-vertex graphs.

Here we construct specs of a particularly nice form. They are optimal for $\widehat{p}(K_n)$ (proved in [2]). We conjecture that they are also optimal for $p(K_n)$, $\widehat{p}(K_{n,n})$, and $p(K_{n,n})$, which is true in some cases.

Definition 3.1 A *canonical coloring* of a graph is an edge-coloring defined by assigning binary vectors (of the same length) as vertex labels and giving each edge the color that is the vector sum of the labels of its endpoints.

Lemma 3.2 *For any graph G, every canonical coloring generated using distinct vertex labels is a spec. If G is bipartite, then every canonical coloring generated from vertex labels such that each is used at most once in each partite set is a spec.*

Proof. Suppose that W is an open walk whose endpoints have names differing in some bit i. The total usage of colors flipping bit i along W must then be odd, and hence some color has odd usage on W. In the bipartite case, we may also have open walks whose endpoints have the same names, but such walks have odd length, which forces odd usage of some color. ■

Corollary 3.3 *If $n = 2^k$, then $\widehat{p}(K_n) = p(K_n) = \chi'(K_n) = n - 1$, and $\widehat{p}(K_{n,n}) = p(K_{n,n}) = \chi'(K_{n,n}) = n$. In general, $\widehat{p}(K_n) \le 2^{\lceil \lg n \rceil} - 1$ and $\widehat{p}(K_{n,n}) \le 2^{\lceil \lg n \rceil}$.*

Proof. The canonical coloring of K_n with colors of length $\lceil \lg n \rceil$ uses $2^{\lceil \lg n \rceil} - 1$ colors (color $\mathbf{0}$ is not used); this equals the trivial lower bound when $n = 2^k$. The same holds for $K_{n,n}$, using color $\mathbf{0}$ also. ■

In [2], we prove that always $\widehat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$. The main idea is to introduce an additional vertex without needing additional colors until a power of 2 is reached. At that point, the trivial lower bound implies that $2^{\lceil \lg n \rceil} - 1$ colors were in use all along. The proof involves studying the vector space $\mathbb{F}_2^k$ of binary k-tuples under component-wise binary addition. A corollary of the proof is that every optimal spec of K_n, for every n, is a canonical coloring generated by vectors of length $\lceil \lg n \rceil$.

Theorem 3.4 ([2]) $\widehat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$. ■

To put this result in perspective and to motivate the conjectures that remain about K_n and $K_{n,n}$, we briefly describe Yuzvinsky's Theorem. Yuzvinsky proved that for subsets A and B of $\mathbb{F}_2^k$ with fixed sizes r and s, the number of vectors that can be obtained as the sum of a vector in A and a vector in B is at least a certain quantity $r \circ s$ called the "Hopf–Stiefel function" of r and s. (In non-algebraic language, $r \circ s$ has an equivalent definition as the least n such that $\binom{n}{k}$ is even for each k with $n - s < k < r$. The condition is vacuous if $n \geq r + s - 1$, so trivially $r \circ s \leq r + s - 1$.)

Later, Plagne computed a nice formula for this function, and Károlyi gave a short proof of that result. We combine these results into a single statement relevant to our context.

Theorem 3.5 (Yuzvinsky [26], Plagne [23], Károlyi [14]) *For $A, B \subseteq \mathbb{F}_2^k$, let $C = \{a + b : a \in A, b \in B\}$. If $|A| = r$ and $|B| = s$, then $|C| \geq r \circ s$, where*

$$r \circ s = \min_{k \in \mathbb{N}} \left\{ 2^k \left(\left\lceil \frac{r}{2^k} \right\rceil + \left\lceil \frac{s}{2^k} \right\rceil - 1 \right) \right\}. \qquad ■$$

When $A = B$, with size r, the minimization yields $r \circ r = 2^{\lceil \lg r \rceil}$. Yuzvinsky's Theorem for this case says that every canonical coloring of K_r uses at least $2^{\lceil \lg r \rceil} - 1$ colors. Our result strengthens this by showing that in the more general family of strong parity edge-colorings, it remains true that at least $2^{\lceil \lg r \rceil} - 1$ colors are needed.

The bound in Yuzvinsky's Theorem is always tight (see [5]); that is, for $r, s \leq 2^k$ there exist $A, B \subseteq \mathbb{F}_2^k$ with $|A| = r$, $|B| = s$, and $|C| = r \circ s$. By Lemma 3.2, $\widehat{p}(K_{r,s}) \leq r \circ s$. We conjecture that equality holds. A direct proof determining $\widehat{p}(K_{r,s})$ in the graph-theoretic setting would strengthen all cases of Yuzvinsky's Theorem.

Conjecture 3.6 $\widehat{p}(K_{r,s}) = r \circ s$.

Yuzvinsky's Theorem as stated describes a bipartite situation, with the application to complete graphs as a special case. This relationship extends to specs, which means that proving the special case of Conjecture 3.6 for $r = s = n$ also implies the result of [2] on K_n. That implication uses the following proposition.

Proposition 3.7 $\widehat{p}(K_n) \geq \widehat{p}(K_{n,n}) - 1$.

Proof. Let f be a spec of K_n with vertex set $u_1, \ldots, u_n$. Given $K_{n,n}$ with partite sets $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$, let $f'(v_i w_j) = f(u_i u_j)$ when $i \neq j$, and give a single new color to all $v_i w_i$ with $1 \leq i \leq n$. A parity walk W' under f' starts and ends in the same partite set. Let W be the walk obtained by mapping it back to K_n, which collapses v_i and w_i into u_i, for each i. The edges that had the new color disappear; this number of edges is even, since W' was a parity walk. Hence W is a parity walk under f.

Since f is a spec, W is a closed walk in K_n. Hence W' starts and ends at vertices in the same partite set that have the same index. Since $K_{n,n}$ has only one vertex with each index in each partite set, W' is closed. Hence f' is a spec of $K_{n,n}$. ■

We have observed that canonical colorings yield $\widehat{p}(K_{n,n}) \leq 2^{\lceil \lg n \rceil}$ for all n. Toward the conjecture that equality holds, we offer the following.

Proposition 3.8 *If some optimal spec of $K_{n,n}$ uses a color on at least $n-1$ edges, then $\widehat{p}(K_{n,n}) = \widehat{p}(K_n) + 1 = 2^{\lceil \lg n \rceil}$. If a color is used $n - r$ times, then $\widehat{p}(K_{n,n}) \geq 2^{\lceil \lg n \rceil} - \binom{r}{2}$.*

Proof. We prove the general statement. Let f be an optimal spec with such a color c. Let U be one part, with $U = u_1, \ldots, u_n$. Whenever u_i or u_j is incident to color c, let $P_{i,j}$ be a u_i, u_j-path of length 2 in which one edge has color c under f. Choose these so that $P_{j,i}$ is the reverse of $P_{i,j}$. When c appears at neither u_i nor u_j, leave $P_{i,j}$ undefined.

Let G be the graph obtained from K_n with vertex set $v_1, \ldots, v_n$ by deleting the edges $v_i v_j$ with $P_{i,j}$ undefined; there are $\binom{r}{2}$ such edges. Define a coloring f' on G by letting $f(v_i v_j)$ be the color other than c on $P_{i,j}$.

We claim that f' is a spec. Given a parity walk W' under f', define a walk W in $K_{n,n}$ as follows. For each edge $v_i v_j$ in W', follow $P_{i,j}$. By construction, each color other than c has even usage in W. Hence also c has even usage. Hence W is a parity walk under f and therefore is closed. Since W starts and ends at the same vertex $u_i \in U$, also W' starts and ends at the same vertex v_i.

We have proved that every parity walk under f' is closed, so f' is a spec. Hence f' has at least $\widehat{p}(G)$ colors, and f has at least one more. By Lemma 2.6(b) and Theorem 3.4, $\widehat{p}(G) \geq 2^{\lceil \lg n \rceil} - 1 - \binom{r}{2}$, which completes the proof of the lower bound.

For the upper bound, Corollary 3.3 shows that $2^{\lceil \lg n \rceil}$ colors suffice. ■

Corollary 3.9 $\widehat{p}(K_{n,n}) \geq \max_r \min\{2^{\lceil \lg n \rceil} - \binom{r}{2}, \frac{n^2}{n-r-1}\}$.

Proof. If $E(K_{n,n})$ has a spec with s colors, where $s < 2^{\lceil \lg n \rceil} - \binom{r}{2}$, then by Proposition 3.8 no color can be used at least $n - r$ times, and hence $n^2/s \le n - r - 1$. Thus $\widehat{p}(K_{n,n}) \ge \min\{2^{\lceil \lg n \rceil} - \binom{r}{2}, n^2/(n - r - 1)\}$. ■

With $r = 1$, we conclude that $\widehat{p}(K_{n,n}) \ge 2^k$ when $n > 2^k - 3 - 4/(n-2)$, since then $n^2/(n-2) > 2^k - 1$. Thus $\widehat{p}(K_{5,5}) = 8$, and $\widehat{p}(K_{n,n}) = 16$ for $13 \le n \le 16$. Using $r = 2$, we obtain $14 \le \widehat{p}(K_{9,9}) \le 16$.

Corollary 3.3 shows that when n is a power of 2, the lower bound of $\Delta(G)$ is optimal for strong parity edge-coloring of $K_{n,n}$. We next enlarge the class of complete bipartite graphs where this bound is optimal.

Theorem 3.10 *If $m = 2^k$ and m divides n, then $p(K_{m,n}) = \widehat{p}(K_{m,n}) = \Delta(K_{m,n}) = n$.*

Proof. Let $r = n/m$ and $[r] = \{1, \dots, r\}$. Label the vertices in the small part with $\mathbb{F}_2^k$. Label those in the large part with $\mathbb{F}_2^k \times [r]$. Color the edges with color set $\mathbb{F}_2^k \times [r]$ by setting $f(uv) = (u + v', j)$, where $v = (v', j)$. In other words, we use r edge-disjoint copies of the bicanonical coloring on r edge-disjoint copies of $K_{m,m}$.

We have used n colors, so it suffices to show that f is a spec. Let W be a parity walk under f. Erasing the second coordinate maps W onto a walk W' in $K_{m,m}$. Furthermore, W' is a parity walk, because all edges in W whose color has the form (z, j) for any j are mapped onto edges with color z under the bicanonical coloring of $K_{m,m}$, and there are an even number of these for each j. Hence W' is closed.

Hence W starts and ends at vertices labeled with the same element u of $\mathbb{F}_2^k$, and they are in the same part since W has even length. If these vertices are different copies of u in the large partite set, then those copies of $K_{m,m}$ have contributed an odd number of edges to W, so for each of them some color confined to it has odd usage in W. This contradicts that W is a parity walk. Hence W is closed, and f is a spec. ■

Corollary 3.11 *If $m \le n$ and $m' = 2^{\lceil \lg m \rceil}$, then $\widehat{p}(K_{m,n}) \le m' \lceil n/m' \rceil$.*

Proof. $K_{m,n} \subseteq K_{m', m'\lceil n/m' \rceil}$. ■

Corollary 3.11 provides examples of complete bipartite graphs where the maximum degree bound is optimal even though the size of neither partite set is a power of 2. For example, $\widehat{p}(K_{3,12}) = 12$. We use Corollary 3.11 next to compute the exact values when $m = 2$. We will apply the result for $K_{2,3}$ in Theorem 4.2.

Corollary 3.12 $\widehat{p}(K_{2,n}) = p(K_{2,n}) = 2 \lceil n/2 \rceil$.

Proof. The upper bound is immediate from Corollary 3.11 with $m' = 2$.

For the lower bound, since $\Delta(K_{2,n}) = n$ for $n \geq 2$, it suffices to show that n must be even when f is a parity edge-coloring of $K_{2,n}$ with n colors. Let $\{x, x'\}$ be the partite set of size 2. Each color appears at both x and x'. If color a appears on xy and $x'y'$, then $f(xy') = f(x'y)$, since otherwise the colors a and $f(xy')$ form a parity path of length 4.

Hence y and y' have the same pair of incident colors. Making this argument for each color partitions the vertices in the partite set of size n into pairs. Hence n is even. ■

The upper bound in Corollary 3.12 can also be proved using an augmentation lemma. If f is a spec of a connected graph G, and G' is formed from G by adding new vertices x and y with common neighbors u and v in G (and no other new edges), then the coloring f' obtained from f by adding two new colors a and b alternating on the new 4-cycle is a spec of G'. This yields $\widehat{p}(G') \leq \widehat{p}(G) + 2$. Like Lemma 2.6(b), this statement fails for disconnected graphs. Since we presently have no further applications for this lemma, we omit the proof.

4 Parity Edge-Coloring of Complete Graphs

Theorem 3.4 states that $\widehat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$, and Conjecture 3.6 asserts that $\widehat{p}(K_{n,n}) = 2^{\lceil \lg n \rceil}$. When n is not a power of 2, these values for K_n and $K_{n,n}$ exceed the maximum degree, which is the trivial lower bound. Hence it is conceivable that in the more general family of parity edge-colorings (not necessarily specs), there is an edge-coloring that uses fewer colors. We conjecture that this is not the case, and that indeed $p(K_n) = \widehat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$, and similarly for complete bipartite graphs.

To prove that $p(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n, it suffices to prove it when n has the form $2^k + 1$. Below we prove it for K_5 and K_9 by case analysis involving counting arguments. This proves the conjecture for K_n whenever $n \leq 16$. Canonical colorings provide the constructions; we only need the lower bounds.

Proposition 4.1 $p(K_5) = 7$.

Proof. Suppose that K_5 has a parity edge-coloring f using at most six colors. Each color class is a matching and hence has size at most 2. Since K_5 has 10 edges, using at most six colors requires at least four classes of size 2. Any two colors used twice must not form a parity path of length 4, so any two colors used twice form an alternating 4-cycle. Hence the colors used twice are all restricted to the same four vertices. However, there are only three disjoint matchings of size 2 in K_4. Thus f cannot exist. ■

Theorem 4.2 $p(K_9) = 15$.

Proof. Let f be a parity edge-coloring using at most 14 colors; we obtain a contradiction. Let C_i be the set of edges in the ith color class, and let $G_{i,j}$ be the spanning subgraph with edge set $C_i \cup C_j$. By Lemma 2.4, a connected subgraph using any k colors has at most 2^k vertices. Hence each $G_{i,j}$ has at least three components. If $|C_i \cup C_j| \geq 7$, then $G_{i,j}$ has at most three components, since the only non-tree components are 4-cycles, allowing the edges to be ordered so that the first six edges reduce the number of components when added.

If each $G_{i,j}$ has at least four components, then $|C_i \cup C_j| \leq 6$. If some class has size 4, then the others have size at most two. Since K_9 has 36 edges, and $4+2\cdot 13 = 30 < 36$, always $|C_i| \leq 3$. However,, $7\cdot 3+7\cdot 2 < 36$, so least eight classes have size 3; let C_i be one of them. If also $|C_j| = 3$, then $G_{i,j}$ has a 4-cycle, since otherwise six edges reduce $G_{i,j}$ to three components. The three edges of C_i form at most six 4-cycles with other colors, but seven other classes have size 3. The contradiction eliminates this case.

Hence we may assume that $G_{1,2}$ has three components A_1, A_2, A_3 with vertex sets V_1, V_2, V_3 and $|V_1| \leq |V_2| \leq |V_3| \leq 4$. Note that $|V_2| \geq 3$. We show that for $i < j$, at least four colors join V_i to V_j. If $|V_j| = 4$, then the edges from V_j to a vertex of V_i have distinct colors. If $|V_j| < 4$, then $|V_j| = 3$ and $|V_i| \geq 2$. The edges joining two vertices of V_i to V_j form $K_{2,3}$. By Corollary 3.12, $p(K_{2,3}) = 4$.

No color class j outside $\{1, 2\}$ connects one of $\{V_1, V_2, V_3\}$ to the other two, since that would yield a connected 9-vertex graph in the three colors $\{1, 2, j\}$, contradicting Corollary 2.4. With three disjoint sets of four colors joining the pairs of components of $G_{1,2}$, we now have 14 colors in f. To avoid using another color, the remaining edges joining vertices within components of $G_{1,2}$ must have colors used joining those components.

Since $|V_2| \geq 3$, we may choose $u, v, w \in V_2$ such that $uv \in C_1$, $vw \in C_2$, and $uw \in C_3$. Let e be an edge of C_3 joining V_i and V_j with $i \neq j$. Suppose first that e is incident to V_2. If $|V_2| = 4$, then $wx \in C_1$ or $ux \in C_2$, and appending e to one end of vu, uw, wx or vw, wu, ux yields a parity path. If $|V_2| = 3$, then $|V_1| \geq 2$, and the end of e other than v is incident to an edge e' in C_1 or C_2. Now e', e, vu, uw or e', e, vw, wu is a parity path.

Hence e joins V_1 and V_3. Let z be the endpoint in V_3. If $|V_3| = 4$, then each of the four colors joining V_2 to V_3 appears at each vertex of V_2. Thus the color on uz is also on some edge wy, and e, zu, uw, wy is a parity path.

Hence $|V_1| = |V_2| = |V_3| = 3$. Since the nine edges joining V_2 and V_3 use only four colors, some color is used on three of the edges. Call it C_4, with edges uu', vv', ww' joining V_2 and V_3. Avoiding a parity path using C_4 with C_1 or C_2 forces $u'v' \in C_1$ and $v'w' \in C_2$. If $z \in \{u', w'\}$, then e, zu, uw, ww' or e, zw, wu, uu' is a parity path. Hence z must be v', and so C_3 appears only once on the copy of $K_{3,3}$ joining V_2 and V_3.

However, $K_{3,3}$ has no parity 4-edge-coloring with a color used only once. The other three colors would have multiplicities $3, 3, 2$. Two matchings of size 3 in $K_{3,3}$ form a 6-cycle, which would contain a parity path. ■

It may be possible to generalize these arguments, but the case analysis seems likely to grow. Instead, we suggest another approach that could help to prove $p(K_n) \geq 2^{\lceil \lg n \rceil} - 1$.

Definition 4.3 A *parity r-set edge-coloring* of a graph G assigns an r-set of colors to each edge of G so that selecting any color from the set on each edge yields a parity edge-coloring of G. Let $p_r(G)$ be the minimum size of the union of the color sets in a parity r-set edge-coloring of G.

Parity r-set edge-coloring is related to parity edge-coloring as r-set coloring is to ordinary proper coloring. An *r-set coloring* of a graph assigns r-sets to the vertices so that the sets on adjacent vertices are disjoint, with $\chi_r(G)$ being the least size of the union of the sets. The *r-set edge-chromatic number* $\chi'_r(G)$ is defined by $\chi'_r(G) = \chi_r(L(G))$. Thus $p_r(G) \geq \chi'_r(G)$.

Using r copies of an optimal parity edge-coloring with disjoint color sets shows that $p_r(G) \leq rp(G)$. We have no examples yet where equality fails. Proving equality could help determine $p(K_n)$ by using the following result.

Proposition 4.4 *If K_n has an optimal parity edge-coloring in which some color class has size $\lfloor n/2 \rfloor$, then $p(K_n) \geq 1 + p_2(K_{\lceil n/2 \rceil})$.*

Proof. Let f be an optimal parity edge-coloring with c used $\lfloor n/2 \rfloor$ times. Let $u_1v_1, \ldots, u_{\lfloor n/2 \rfloor}v_{\lfloor n/2 \rfloor}$ be the edges with color c, and let $u_{\lceil n/2 \rceil}$ be the vertex missed by c if n is odd. Contracting these edges yields $K_{\lceil n/2 \rceil}$, with u_iv_i contracting to w_i for $i \leq \lfloor n/2 \rfloor$, and $w_{\lceil n/2 \rceil} = u_{\lceil n/2 \rceil}$ when n is odd.

For each edge w_iw_j in $K_{\lceil n/2 \rceil}$, with $i < j \leq \lceil n/2 \rceil$, define $f'(w_iw_j) = \{f(u_iu_j), f(v_iu_j)\}$. Since f' does not use c, to prove $p_2(K_{\lceil n/2 \rceil}) \leq p(K_n) - 1$ it suffices to show that f' is a parity 2-set edge-coloring.

If f' is not a parity 2-set edge-coloring, then some selection of edge colors from f' forms a parity path P'. Form a path P in K_n as follows. When P' follows the edge w_iw_j with chosen color a, P moves along the edge u_iv_i of color c (if necessary) to reach an endpoint in $\{u_i, v_i\}$ of an edge with color a under f whose other endpoint is in $\{u_j, v_j\}$, and then it follows that edge. This path has the same usage as P' for every color other than c. Since c misses only one vertex of K_n, at least one end of P' is a contracted vertex, and an edge of color c can be added or deleted at that end of P to make the usage of c even if it had been odd. If P' is a w_i, w_j-path, then P starts in $\{u_i, v_i\}$ and ends in $\{u_j, v_j\}$ (one of the sets may degenerate to $\{u_{\lceil n/2 \rceil}\}$). Now P is a parity path under f, which is a contradiction. ■

If $n = 2^k+1$, then $\lceil n/2 \rceil = 2^{k-1}+1$. If there is always an optimal parity edge-coloring of K_n with a near-perfect matching, then proving $p_2(K_n) = 2p(K_n)$ would inductively prove that $p(K_n) = 2^{\lceil \lg n \rceil} - 1$. Although we do not know whether $p_2(G) = 2p(G)$ in general, we provide support for the various conjectures by proving this when G is a path.

Theorem 4.5 *$p_r(P_n) = rp(P_n)$.*

Proof. We prove the stronger statement that for every parity r-set edge-coloring f of P_n, some set of $p(P_n)$ edges has pairwise disjoint color sets.

Let $e_1, \ldots, e_{n-1}$ be the edge set of P_n in order. We say that a subset $\{e_{i_1}, \ldots, e_{i_q}\}$ of $E(P_n)$ with $i_1 < \cdots < i_q$ is *linked* by f if $f(e_{i_j}) \cap f(e_{i_{j+1}}) \neq \varnothing$ for $1 \leq j \leq q-1$.

Suppose that $E(P_n)$ decomposes into linked sets $S_1, \ldots, S_t$ under f. We show that setting $f'(e) = i$ when $e \in S_i$ yields a parity t-edge-coloring f' of P_n. If not, then let Q be a parity path under f'. Since Q has even usage of color i, we can pair successive edges among those having color i (first with second, third with fourth, etc.). Since S_i is linked, the two sets assigned to a pair have a common color. Picking this for each pair and each color under f' selects colors from the sets assigned to Q under f that form a parity path. This contradicts the choice of f as a parity r-set edge-coloring. Thus every partition of $E(P_n)$ into linked sets needs at least $p(P_n)$ parts.

To obtain edges with disjoint color sets from such a partition, first construct a bipartite graph H with partite sets $v_1, \ldots, v_{n-1}$ and $w_1, \ldots, w_{n-1}$ by letting v_iw_j be an edge if and only if $i < j$ and $f(e_i) \cap f(e_j) \neq \varnothing$. If $E(P_n)$ has a partition into t linked sets, then H has a matching of size $n-1-t$, obtained by using the edge v_iw_j when e_i and e_j are successive elements in one part of the partition.

The construction of a matching from a partition is reversible. As edges are added to the matching, starting from the empty matching and the partition into singletons, the structural property is maintained that for the edges in a part, only the first edge e_j has w_j unmatched, and only the last edge e_i has v_i unmatched. Hence when an edge v_iw_j is added to the matching, it links the end of one part to the beginning of another part, reduces the number of parts, and maintains the structural property.

Thus $E(P_n)$ has a partition into t linked sets under f if and only if H has a matching of size $n-1-t$. When t is minimized, the König–Egerváry Theorem yields a vertex cover of H with size $n-t-1$. Because the complement of a vertex cover is an independent set, H has an independent set T of size $n+t-1$. Since $V(H)$ consists of $n-1$ pairs of the form $\{v_i, w_i\}$, at least t such pairs are contained in T. If $\{v_i, w_i\}, \{v_j, w_j\} \subseteq T$, then $f(e_i)$ and $f(e_j)$ are disjoint. Therefore there is a set of t edges whose color sets are pairwise disjoint.

We conclude that $p_r(P_n) \geq rt \geq rp(P_n)$. ∎

5 Other Related Edge-Coloring Parameters

In this section we describe other parameters defined by looser or more restricted versions of parity edge-coloring, and we give examples to show that $p(G)$ is a different parameter.

A *nonrepetitive edge-coloring* is an edge-coloring in which no pattern repeats immediately on a path. That is, no path may have colors $c_1, \ldots, c_k$ followed immediately by $c_1, \ldots, c_k$ in order, for any k. The notion was introduced for graphs in [1]. Every parity edge-coloring is nonrepetitive, and every nonrepetitive edge-coloring is proper, so the minimum number of colors in a non-repetitive edge-coloring of G lies between $p(G)$ and $\chi'(G)$. The resulting parameter is called the *Thue chromatic number* in honor of the famous theorem of Thue constructing non-repetitive sequences (generalized to graphs in [1]). The concept is surveyed in [7].

More restricted versions of parity edge-colorings have also been studied. A *conflict-free coloring* is an edge-coloring in which every path uses some color exactly once. An *edge-ranking* is an edge-coloring in which on every path, the highest-indexed color appears exactly once. Letting $c(G)$ and $t(G)$ denote the minimum numbers of colors in a conflict-free coloring and an edge-ranking, respectively, we have $t(G) \geq c(G) \geq p(G)$.

Conflict-free coloring has been studied primarily in geometric settings; see [6, 8, 22]. Edge-rankings were introduced in [13]. It is known that $t(K_n) \in \Omega(n^2)$ [3]; since $p(K_n) \leq 2n - 3$, the gap here can be large. Equality can hold: $t(P_n) = c(P_n) = p(P_n) = \lceil \lg n \rceil$. Although computing $p(G)$ or $\widehat{p}(G)$ is NP-hard when G is restricted to trees, there is a algorithm to compute $t(G)$ that runs in linear time when G is a tree [18] (at least four slower polynomial-time algorithms were published earlier). Computing $t(G)$ is NP-hard on general graphs [17], as is finding a spanning tree T with minimal $t(T)$ [19].

In this string of inequalities, $c(G)$ and $p(G)$ are neighboring parameters. In this section, we present examples to show that they may differ. In fact, in all these examples $c(G) > \widehat{p}(G)$.

Corollary 5.1 *$c(K_{2^k}) > \widehat{p}(K_{2^k}) = p(K_{2^k})$ when $k \geq 4$.*

Proof. We have observed that the proof of Theorem 3.4 in [2] implies that every optimal spec f of K_{2^k} is canonical. The spanning subgraph of K_{2^k} formed by the color classes whose names are vectors of weight 1 is isomorphic to the hypercube Q_k, and the colors on it correspond to the coordinate directions. If there is a path in Q_k that crosses each coordinate direction more than once, then f is not conflict-free and $c(K_{2^k}) > \widehat{p}(K_{2^k})$. In fact, it is easy to find such paths when $k \geq 4$. ∎

Example 5.2 As noted in Corollary 2.5, $\widehat{p}(C_8) = 3$. Suppose that C_8 has a conflict-free 3-edge-coloring. If a color is used only once, then the other two colors alternate on paths of length 4 avoiding it, thus forming parity paths of length 4. Hence the sizes of the three color classes must be $(4,2,2)$ or $(3,3,2)$. Now deleting a edge from a largest color class yields a spanning path on which no color appears only once.

By induction on the length, every path has an optimal parity edge-coloring that is conflict-free (use a color only on a middle edge and apply the induction hypothesis to each component obtained by deleting that edge). This statement does not hold for trees.

Definition 5.3 A *broom* is a tree formed by identifying an endpoint of a path with a vertex of a star. Let T_k be the broom formed using P_{2^k-2k+2} and a leaf of a star with k edges. The *parity* of a vertex in Q_k is the parity of the weight of the k-tuple naming it.

We prove that T_k embeds in Q_k but needs more than k colors for a conflict-free edge-coloring (for $k \geq 4$). W. Kinnersley (private communication) showed this initially for $k = 5$. We must first show that T_k indeed embeds in Q_k. This follows from the result of [15] that "double-star-like" tree with 2^{k-1} vertices in each partite set embed as spanning trees of Q_k, since adding $k-2$ leaf neighbors to the 2-valent neighbor of the k-valent vertex in T_k yields such a tree. Their proof is lengthy; we give a short direct proof for this special case.

Lemma 5.4 *If x and y are distinct vertices of Q_k having the same parity, then there is a path of length $2^k - 3$ in Q_k that starts at x and avoids y.*

Proof. It is well known that Q_k has a spanning cycle when $k \geq 2$. Since Q_k is edge-transitive, there is a spanning path from each vertex to any adjacent vertex (for $k \geq 1$).

The desired path exists by inspection when $k = 2$. For larger k, we proceed inductively. Vertices x and y differ in an even number of bits; by symmetry, we may assume that they differ in the first two bits. Let Q' and Q'' be the $(k-1)$-dimensional subcubes induced by the vertices with first bit 0 and first bit 1, respectively. We may assume that $x \in V(Q')$. There is a spanning x,u-path P' of Q', where u is the neighbor of x obtained by changing the third bit. Note that P' has length $2^{k-1} - 1$.

Let v be the neighbor of u in Q''. Since v has the same parity as y, and $v \neq y$, the induction hypothesis yields a path P'' of length $2^{k-1} - 3$ in Q'' that starts at v and avoids y. Together, P', uv, and P'' complete the desired path in Q_k. ■

Lemma 5.5 *For $k \geq 2$, the broom T_k embeds in Q_k, and hence $\widehat{p}(T_k) = p(T_k) = k$.*

Proof. Note that $T_k = P_4 \subseteq Q_k$ when $k = 2$; we proceed inductively. For $k > 2$, the tree T_k contains T_{k-1}, obtained by deleting one leaf incident to the vertex v of degree k and $2^{k-1} - 2$ vertices from the other end. With Q' and Q'' defined in Lemma 5.4, by the induction hypothesis T_{k-1} embeds in Q'. The distance in T_{k-1} from v to its leaf nonneighbor u is $2^{k-1} - 2(k-1) + 2$. This is even, so u and v have the same parity. Let x and y be the neighbors of u and v in Q'', respectively; also x and y have the same parity. By Lemma 5.4, Q'' contains a path P of length $2^{k-1} - 3$ starting from x and avoiding y. Now adding vy, ux, and P to the embedding of T_{k-1} yields the desired embedding of T_k in Q_k. ■

Theorem 5.6 *If $k \geq 4$, then $c(T_k) = k + 1 = \widehat{p}(T_k) + 1$.*

Proof. For $k = 4$, a somewhat lengthy case analysis is needed to show $c(T_4) > 4$; we omit this. Let x be the vertex of degree k in T_k.

For $k \geq 5$, we decompose T_k into several pieces. At one end is a star S with $k-1$ leaves and center x. Let P be the path of length 2^{k-2} beginning with x. Let R be the path of length 2^{k-1} beginning at the other end of P. Since $k \geq 5$, we have $2^{k-2} + 2^{k-1} \leq 2^k - 2k + 2$, so P and R fit along the handle of the broom. Ignore the rest of T_k after the end of R.

Consider a conflict-free k-edge-coloring of $S \cup P \cup R$. Since R has $2^{k-1}+1$ vertices, at least k colors appear on $E(R)$. Since P has $2^{k-2}+1$ vertices, at least $k-1$ colors appear on $E(P)$. Hence on $P \cup R$ there are $k-1$ colors that appear at least twice, and only one color c appears exactly once. Since x has degree k, all k colors appear incident to x, including c. Hence c appears on some edge of S, and adding this edge to $P \cup R$ yields a path on which every color appears at least twice.

For $k \geq 2$, we obtain a conflict-free $(k+1)$-edge-coloring using color $k+1$ only on the edge e of P at x. Deleting e leaves the star S and a path P' with $2^k - 2k + 2$ vertices. Since S has $k-1$ edges and the length of P' is less than 2^k, each has a conflict-free edge-coloring using colors 1 through k. Paths from $V(S)$ to $V(P')$ use color $k+1$ exactly once. ■

It remains unknown how large $c(G)$ can be when $\widehat{p}(G) = k$ or $p(G) = k$, either in general or when G is restricted to be a tree.

6 Open Problems

Many interesting questions remain about parity edge-coloring and strong parity edge-coloring. We have already mentioned several and collect them here with additional questions.

In Section 4, we proved the first conjecture for $n \leq 16$. In Section 3, we proved various special cases of the second conjecture, which yield further special cases of the first.

Conjecture 6.1 $p(K_n) = 2^{\lceil \lg n \rceil} - 1$ *for all* n.

Conjecture 6.2 $p(K_{n,n}) = \widehat{p}(K_{n,n}) = 2^{\lceil \lg n \rceil}$ *for all* n.

For complete bipartite graphs in general, the full story would be given by proving Conjecture 3.6, which we restate here for completeness.

Conjecture 6.3 $\widehat{p}(K_{r,s}) = r \circ s$ *for all* r *and* s.

We have exhibited families of graphs G such that $\widehat{p}(G) > p(G)$ (see Example 2.8), but the difference is only 1, and the graphs we obtained all contain odd cycles.

Question 6.4 What is the maximum of $\widehat{p}(G)$ when $p(G) = k$?

Conjecture 6.5 $p(G) = \widehat{p}(G)$ *for every bipartite graph* G.

If Conjecture 6.5 holds, than Conjecture 6.3 also determines $p(K_{r,s})$. If the conjectures are not both true, then it would still be interesting to know how $p(K_{k,n})$ and $\widehat{p}(K_{k,n})$ grow with k for fixed n. In particular, when do they reach $2^{\lceil \lg n \rceil}$? Theorem 3.10 may shed some light. Does equality hold in Corollary 3.11?

Several questions about parity edge-coloring of trees are related to which trees with 2^k vertices embed as spanning trees of Q_k. Havel [9] proposed studying that question, and many papers followed; Havel [10] presents a survey. Such a tree must have partite sets of equal size and must have maximum degree at most k, but these conditions are not sufficient. Later results on sufficient conditions include [4, 11, 15, 21].

Question 6.6 What is the maximum of $p(T)$ among n-vertex trees T with $\Delta(T) = D$?

We observed from Theorem 2.2 that testing $p(T) \leq k$ is NP-hard. This suggests complexity questions for more restricted problems.

Question 6.7 Do polynomial-time algorithms exist for computing $p(T)$ on trees with maximum degree D or on trees with bounded diameter?

The algebraic arguments in [2] yield that recognition of specs is in P. However, we do not know whether the same holds for parity edge-coloring. (It does hold for edge-colorings of trees using the labeling procedure of Theorem 2.2.)

Question 6.8 What is the complexity of testing whether an edge-coloring is a pec?

Paths and complete graphs show that $p(G)$ is unbounded for fixed maximum degree or diameter. However, bounding both parameters limits the number of vertices. Hence the next question makes sense.

Question 6.9 What is the maximum of $p(G)$ among graphs (or trees) with $\Delta(G) \leq k$ and $\operatorname{diam}(G) \leq d$?

It is a classical question to determine the maximum number of edges in an n-vertex subgraph of Q_k, where $n \leq 2^k$. Does the resulting graph have the maximum number of edges in an n-vertex graph with parity edge-chromatic number k? More generally,

Question 6.10 What is the minimum of $p(G)$ among all n-vertex graphs having m edges?

The lower bound in Corollary 2.4 naturally leads us to ask which graphs achieve equality. Every spanning subgraph of a hypercube satisfies $p(G) = \lg n(G)$; is the converse true?

Question 6.11 For which connected graphs G is it true that $p(G) = \lceil \lg n(G) \rceil$? Which satisfy $\widehat{p}(G) = \lceil \lg n(G) \rceil$?

Motivated by the uniqueness of the optimal spec of K_{2^k}, Dhruv Mubayi suggested studying the "stability" of the result.

Question 6.12 Does there exist an parity edge-coloring of K_{2^k} with $(1 + o(1))2^k$ colors that is "far" from the canonical coloring?

In Section 5, we showed that paths satisfy all three properties below. Are there other such graphs?

Question 6.13 For which graphs G do the following (successively stronger) properties hold? (a) $p_2(G) = 2p(G)$?
(b) $p_r(G) = rp(G)$ for all r?
(c) every parity r-set edge-coloring of G contains a set of $p(G)$ edges whose color sets are pairwise disjoint?

Lemma 2.6(a) states that deleting an edge reduces the parity edge-chromatic number by at most 1. Ordinary coloring has the same property. Thus we are motivated to call a graph G *critical* if $p(G - e) < p(G)$ for all $e \in E(G)$. We say that G is *doubly-critical* if $p(G - e - e') = p(G) - 2$ for all $e, e' \in E(G)$. Our results on paths and cycles imply that for all $n \geq 1$, P_{2^n+1} is critical and C_{2^n+1} is doubly-critical. Naturally, any star is doubly-critical.

Question 6.14 Which graphs are critical? Which graphs are doubly-critical?

Since the factors can be treated independently in constructing a spec, $\widehat{p}$ is subadditive under Cartesian product. Note that $\widehat{p}(P_2 \Box P_2) = 2 = \widehat{p}(P_2) + \widehat{p}(P_2)$.

Question 6.15 For what graphs G and H does equality hold in $\widehat{p}(G \Box H) \leq \widehat{p}(G) + \widehat{p}(H)$? What can be said about $p(G \Box H)$ in terms of $p(G)$ and $p(H)$?

It may be interesting to compare $p(G)$ with related parameters such as conflict-free edge-chromatic number on special classes of graphs. We suggest two specific questions.

Question 6.16 What is the maximum of $c(T)$ such that T is a tree with $p(T) = k$? What is the maximum among all graphs with parity edge-chromatic number k?

Finally, the definitions of parity edge-coloring and spec extend naturally to directed graphs: the parity condition is the same but is required only for directed paths or walks. Hence $p(D) \leq p(G)$ and $\widehat{p}(D) \leq \widehat{p}(G)$ when D is an orientation of G.

For a directed path $\vec{P}_m$, the constraints are the same as for an undirected path. More generally, if D is an acyclic digraph, and m is the maximum number of vertices in a path in D, then $p(D) = \widehat{p}(D) = \lceil \lg m \rceil$. The lower bound is from any longest path.

For the upper bound, give each vertex x a label $l(x)$ that is the maximum number of vertices in a path ending at x (sources have label 0). Write each label as a binary $\lceil \lg m \rceil$-tuple. By construction, $l(v) > l(u)$ whenever uv is an edge. To form a spec of D, use a color c_i on edge uv if the ith bit is the first bit where $l(u)$ and $l(v)$ differ. All walks are paths. Any x, y-path has odd usage of c_i, where the ith is the first bit where $l(x)$ and $l(y)$ differ, since no edge along the path can change an earlier bit.

Thus the parameters equal $\lceil \lg n \rceil$ for the n-vertex transitive tournament, which contains $\vec{P}_n$. This suggests our final question.

Question 6.17 What is the maximum of $p(T)$ or $\widehat{p}(T)$ when T is an n-vertex tournament?

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